

Fourier Series and Integral:

$$f(t) = \frac{a_o}{2} + a_1 \cos \frac{\pi t}{p} + a_2 \cos \frac{2\pi t}{p} + a_3 \cos \frac{3\pi t}{p} + \dots + a_n \cos \frac{n\pi t}{p} \\ b_1 \sin \frac{\pi t}{p} + b_2 \sin \frac{2\pi t}{p} + b_3 \sin \frac{3\pi t}{p} + \dots + b_n \sin \frac{n\pi t}{p}$$

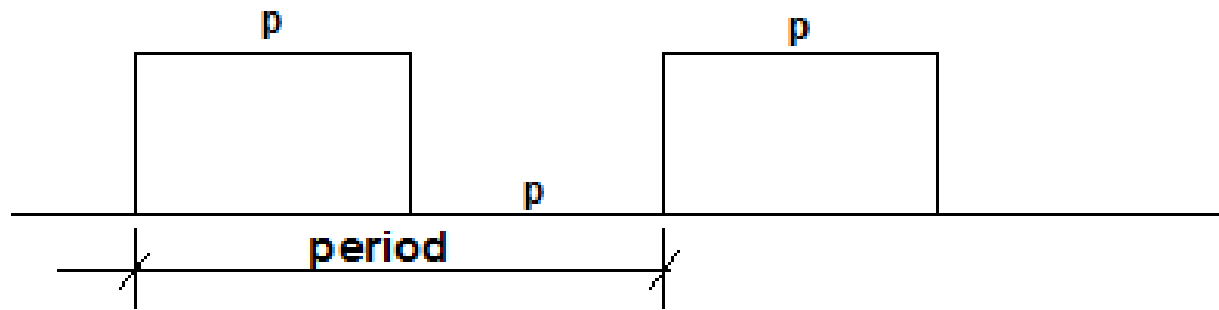
$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p} \right)$$

$$a_o = \frac{1}{p} \int_d^{d+2p} f(t) dt$$

$$a_n = \frac{1}{p} \int_d^{d+2p} f(t) \cos \frac{n\pi t}{p} dt$$

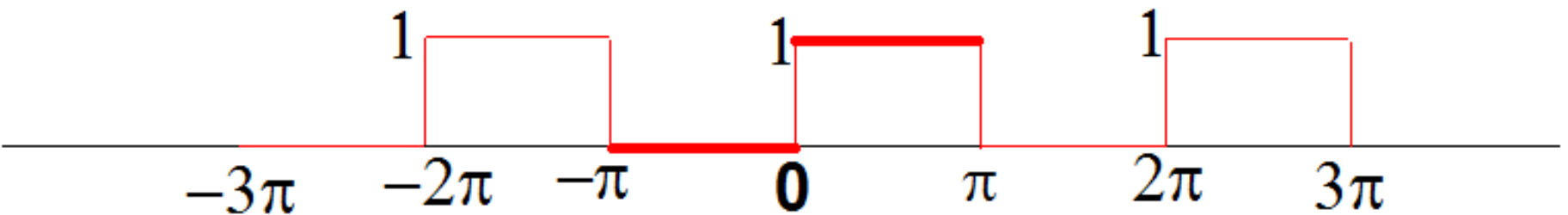
$$b_n = \frac{1}{p} \int_d^{d+2p} f(t) \sin \frac{n\pi t}{p} dt$$

$2p = \text{period}$



Example:

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$



Solution:

$$p = \pi$$

$$a_o = \frac{1}{p} \int_d^{d+2p} f(t) dt$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 0 dx + \int_0^{\pi} 1 dx \right] = \frac{1}{\pi} * \pi = 1$$

$$a_n = \frac{1}{p} \int_d^{d+2p} f(t) \cos \frac{n\pi t}{p} dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 0 \cos nx dx + \int_0^{\pi} 1 \cos nx dx \right]$$

$$= \frac{1}{\pi} \int_0^{\pi} \cos nx dx = \frac{1}{\pi} \frac{1}{n} \sin nx \Big|_0^{\pi} = 0 \quad n \neq 0$$

$$\begin{aligned}
b_n &= \frac{1}{p} \int_d^{d+2p} f(t) \sin \frac{n\pi t}{p} dt \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\
&= \frac{1}{\pi} \left[\int_{-\pi}^0 0 \sin nx dx + \int_0^{\pi} 1 \sin nx dx \right] \\
&= \frac{1}{\pi} \int_0^{\pi} \sin nx dx = \frac{1}{\pi} \frac{1}{n} (-\cos nx) \Big|_0^{\pi} \\
&= \frac{1}{n\pi} (-\cos n\pi - (-\cos 0)) = \frac{1}{n\pi} (1 - \cos n\pi) \\
&= \frac{1}{n\pi} (1 - (-1)^n) = \begin{cases} 0 & n = \text{even} \\ \frac{2}{n\pi} & n = \text{odd} \end{cases} \\
\therefore b_n &= \frac{2}{n\pi} \quad \text{for } n = \text{odd}
\end{aligned}$$

$$a_o = 1$$

$$a_n = 0$$

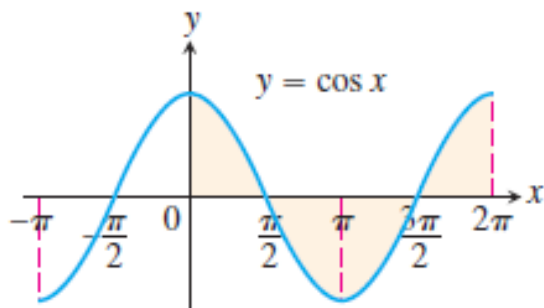
$$b_n = \frac{2}{n\pi} \quad \text{for } n = \text{odd}$$

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right]$$

$$= \frac{1}{2} + \sum_{n=1,3,5}^n \frac{2}{n\pi} \sin \frac{n\pi x}{\pi}$$

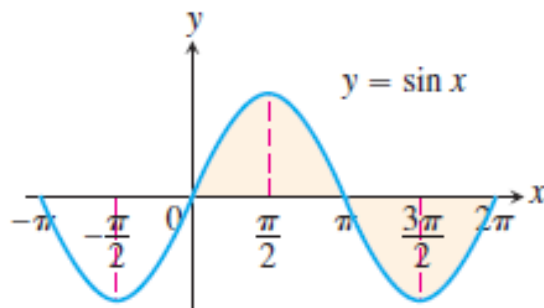
$$= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1,3,5}^n \frac{1}{n} \sin nx$$

$$\therefore f(x) = \frac{1}{2} + \frac{2}{\pi} \left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \dots \right]$$



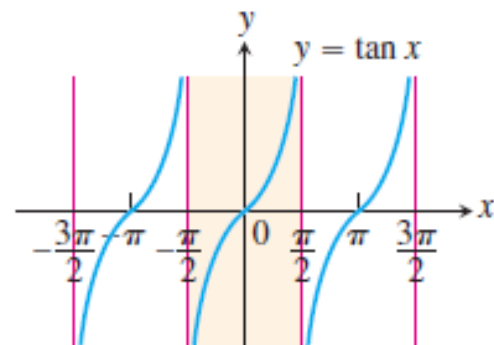
Domain: $-\infty < x < \infty$
 Range: $-1 \leq y \leq 1$
 Period: 2π

(a)



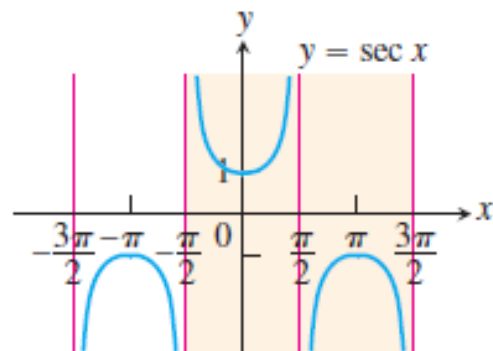
Domain: $-\infty < x < \infty$
 Range: $-1 \leq y \leq 1$
 Period: 2π

(b)



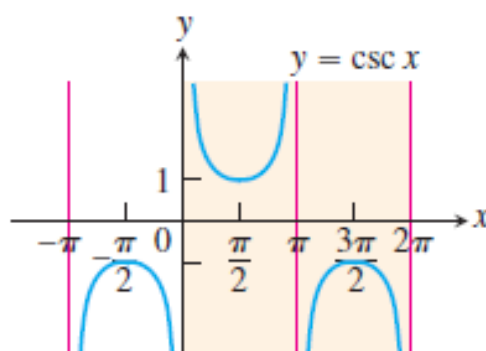
Domain: $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$
 Range: $-\infty < y < \infty$
 Period: π

(c)



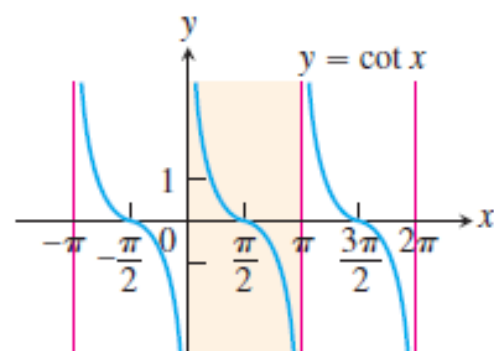
Domain: $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$
 Range: $y \leq -1$ or $y \geq 1$
 Period: 2π

(d)



Domain: $x \neq 0, \pm\pi, \pm2\pi, \dots$
 Range: $y \leq -1$ or $y \geq 1$
 Period: 2π

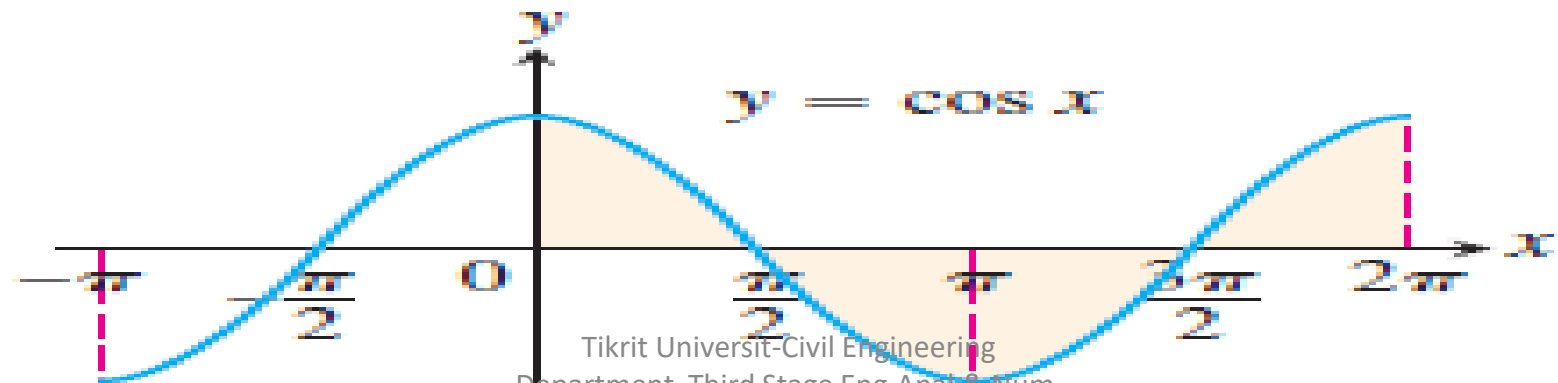
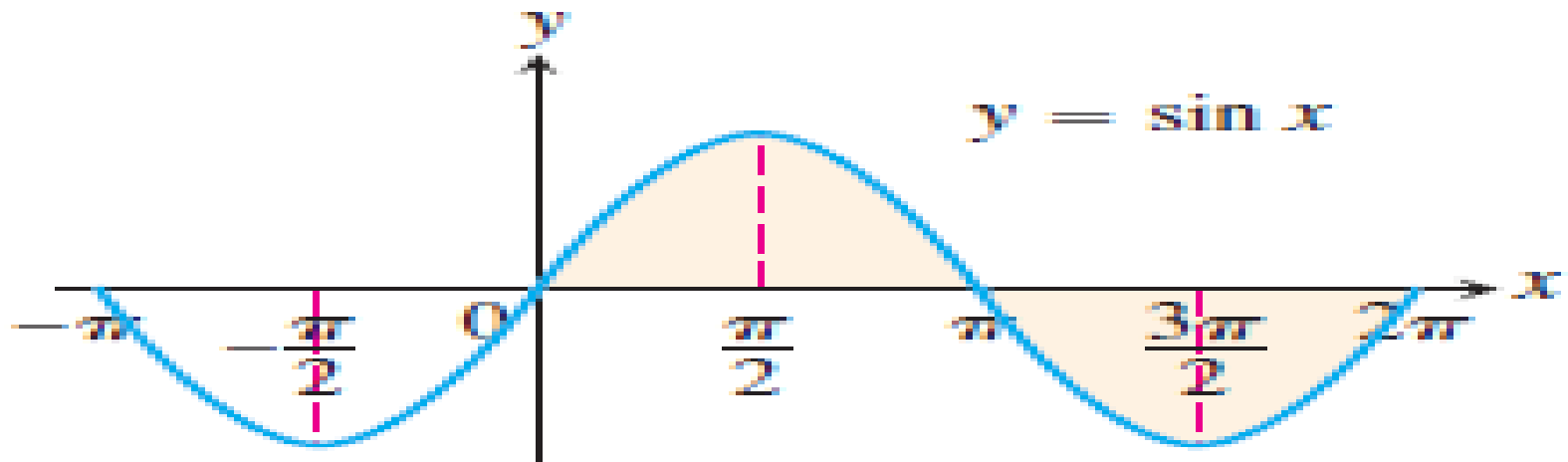
(e)



Domain: $x \neq 0, \pm\pi, \pm2\pi, \dots$
 Range: $-\infty < y < \infty$
 Period: π

(f)

To obtain formula the coefficients a_n and b_n , we need the following integrals, where m and n are integers.



$$1 - \int_d^{d+2p} \cos \frac{n\pi t}{p} dt = 0 \quad n \neq 0$$

$$2 - \int_d^{d+2p} \sin \frac{n\pi t}{p} dt = 0$$

$$3 - \int_d^{d+2p} \cos \frac{m\pi t}{p} \sin \frac{n\pi t}{p} dt = 0 \quad m \neq n$$

$$4 - \int_d^{d+2p} \cos^2 \frac{m\pi t}{p} dt = p \quad n \neq 0$$

$$5 - \int_d^{d+2p} \cos \frac{m\pi t}{p} \sin \frac{n\pi t}{p} dt = 0 \quad m = n$$

$$6 - \int_d^{d+2p} \sin \frac{m\pi t}{p} \sin \frac{n\pi t}{p} dt = 0 \quad m \neq n$$

$$7 - \int_d^{d+2p} \sin^2 \frac{n\pi t}{p} dt = p \quad n \neq 0$$

Limits :

$$1 - \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

$$2 - \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

$$3 - \lim_{n \rightarrow \infty} x^{\frac{1}{n}} = 1 \quad (x > 0)$$

$$4 - \lim_{n \rightarrow \infty} x^n = 0 \quad (|x| < 1)$$

$$5 - \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

$$6 - \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$$

Integrals:

(m and n are integers)

$$1 - \int_0^{2\pi} \sin nx \, dx = \int_0^{2\pi} \cos nx \, dx = 0$$

$$2 - \int_0^{2\pi} \sin mx \sin nx \, dx = \frac{1}{2} \int_0^{2\pi} [\cos (m-n)x - \cos(m+n)x] \, dx = 0$$

$$3 - \int_0^{2\pi} \sin^2 nx \, dx = \frac{1}{2} \int_0^{2\pi} (1 - \cos 2nx) \, dx = \pi$$

$$4 - \int_0^{2\pi} \cos mx \cos nx \, dx = 0$$

$$5 - \int_0^{2\pi} \cos mx \sin nx \, dx = 0$$

$$6 - \int_0^{2\pi} \cos^2 nx \, dx = 0$$

$$7 - \int_0^{2\pi} \sin nx \cos nx \, dx = \frac{1}{2} \int_0^{2\pi} \sin 2nx \, dx = 0$$

$$8 - \int \sin ax \cos bx \, dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} + c \quad a^2 \neq b^2$$

$$9 - \int \sin ax \sin bx \, dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + c \quad a^2 \neq b^2$$

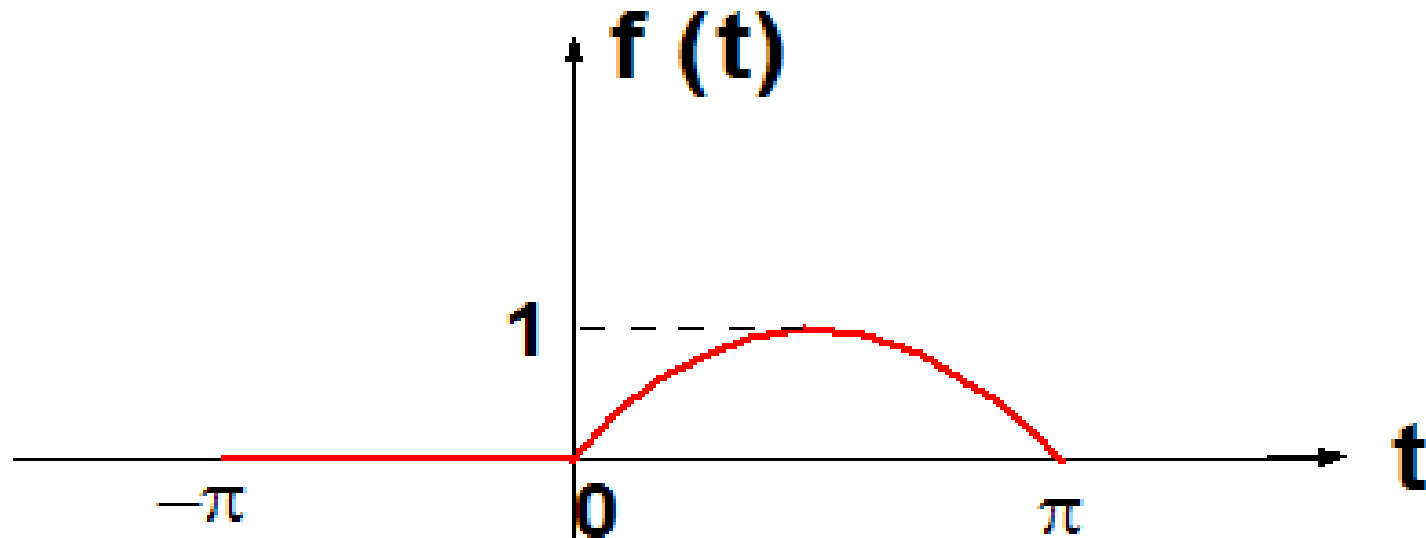
$$10 - \int \cos ax \cos bx \, dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + c \quad a^2 \neq b^2$$

$$11 - \int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin ax}{4a} + c$$

$$12 - \int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin ax}{4a} + c$$

Example: What is the Fourier expansion of the periodic function whose definition in one period is:

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ \sin t & 0 < t < \pi \end{cases}$$



Solution:

$$a_o = \frac{1}{p} \int_d^{d+2p} f(t) dt$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 0 dt + \int_0^{\pi} \sin t dt \right] = \frac{1}{\pi} \left(-\cos t \Big|_0^{\pi} \right)$$

$$= \frac{1}{\pi} [-(\cos \pi - \cos 0)] = \frac{1}{\pi} [-(-1 - 1)] = \frac{2}{\pi}$$

$$\begin{aligned}
 a_n &= \frac{1}{p} \int_d^{d+2p} f(t) \cos \frac{n\pi t}{p} dt \\
 &= \frac{1}{\pi} \left[\int_{-\pi}^0 0 \cos \frac{n\pi t}{\pi} dt + \int_0^{\pi} \sin t \cos \frac{n\pi t}{\pi} dt \right] \\
 &= \frac{1}{\pi} \int_0^{\pi} \sin t \cos nt dt \\
 &= \frac{1}{\pi} \left[-\frac{\cos(1-n)t}{2(1-n)} - \frac{\cos(1+n)t}{2(1+n)} \right]_0^{\pi} \\
 &= -\frac{1}{2\pi} \left[\left(\frac{\cos(\pi - n\pi)}{1-n} + \frac{\cos(\pi + n\pi)}{1+n} \right) - \left(\frac{1}{1-n} + \frac{1}{1+n} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 a_n &= -\frac{1}{2\pi} \left[\left(\frac{\cos(\pi - n\pi)}{1-n} + \frac{\cos(\pi + n\pi)}{1+n} \right) - \left(\frac{1}{1-n} + \frac{1}{1+n} \right) \right] \\
 &= -\frac{1}{2\pi} \left(\frac{-\cos n\pi}{1-n} + \frac{-\cos n\pi}{1+n} - \frac{2}{1-n^2} \right) \\
 &= -\frac{1}{2\pi} \left(\frac{-2\cos n\pi}{1-n^2} - \frac{2}{1-n^2} \right) \\
 &= \frac{1 + \cos n\pi}{\pi(1-n^2)} \quad n \neq 1 \quad (a_n = 0 \quad n = \text{odd})
 \end{aligned}$$

$$a_1 = \frac{1}{\pi} \int_0^\pi \sin t \cos t \, dt = \frac{\sin^2 t}{2\pi} \Big|_0^\pi = 0$$

$$\begin{aligned}
 b_n &= \frac{1}{p} \int_d^{d+2p} f(t) \sin \frac{n\pi t}{p} dt \\
 &= \frac{1}{\pi} \left[\int_{-\pi}^0 0 \sin \frac{n\pi t}{\pi} dt + \int_0^{\pi} \sin t \sin \frac{n\pi t}{\pi} dt \right] \\
 &= \frac{1}{\pi} \int_0^{\pi} \sin t \sin nt dt \\
 &= \frac{1}{\pi} \left[\frac{\sin(1-n)t}{2(1-n)} - \frac{\sin(1+n)t}{2(1+n)} \right]_0^{\pi} \\
 &= \frac{1}{2\pi} \left[\frac{\sin(\pi - n\pi)}{1-n} + \frac{\sin(\pi + n\pi)}{1+n} \right]
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{1}{2\pi} \left[\frac{\sin(\pi - n\pi)}{1-n} + \frac{\sin(\pi + n\pi)}{1+n} \right] \\
 &= \frac{1}{2\pi} \left(\frac{\sin n\pi}{1-n} - \frac{-\sin n\pi}{1+n} \right) \\
 &= \frac{\sin n\pi}{\pi(1-n^2)} = 0 \quad n \neq 1
 \end{aligned}$$

$$\begin{aligned}
 b_1 &= \frac{1}{\pi} \int_0^\pi \sin^2 t \, dt = \frac{1}{\pi} \int_0^\pi \left(\frac{1 - \cos 2t}{2} \right) dt \\
 &= \frac{1}{\pi} \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^\pi = \frac{1}{\pi} \frac{\pi}{2} = \frac{1}{2}
 \end{aligned}$$

$$a_o = \frac{2}{\pi}$$

$$a_n = \begin{cases} 0 & n = \text{odd} \\ \frac{1 + \cos n \pi}{\pi(1 - n^2)} & n = \text{even} \end{cases} \quad (n \neq 1)$$

$$a_1 = 0$$

$$b_n = 0 \quad (n \neq 1)$$

$$b_1 = \frac{1}{2}$$

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n \pi t}{p} + b_n \sin \frac{n \pi t}{p} \right]$$

$$\therefore f(t) = \frac{1}{\pi} + \frac{\sin t}{2} - \frac{2}{\pi} \left[\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \dots \right]$$

Note :

$$\cos(\pi - n\pi) = \cos \pi \cos n\pi + \sin \pi \sin n\pi = -\cos n\pi$$

$$\cos(\pi + n\pi) = \cos \pi \cos n\pi - \sin \pi \sin n\pi = -\cos n\pi$$

Numerical Series can be obtained from Fourier series,

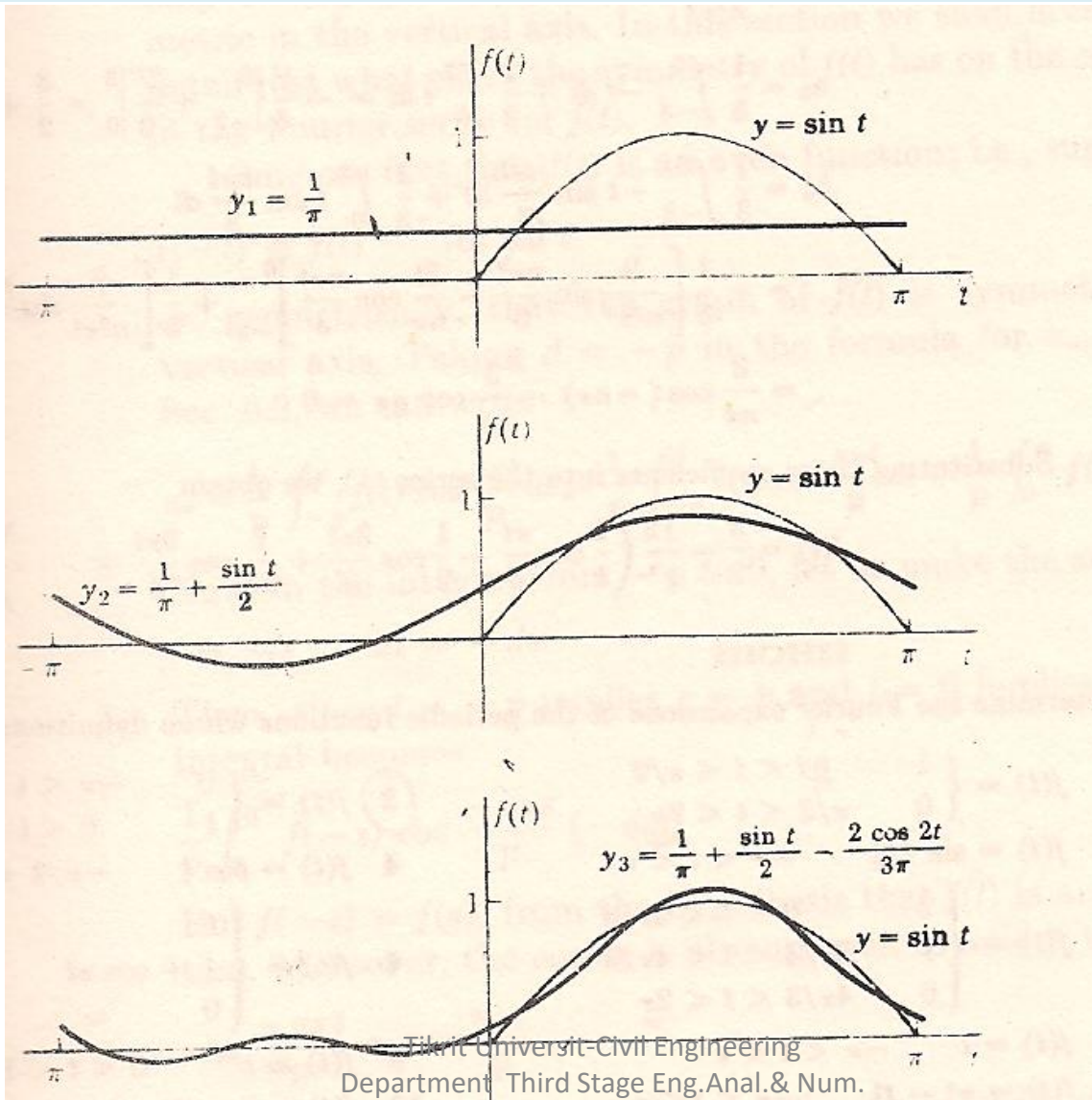
for example: If $t = \frac{\pi}{2}$ in the previous example:

$$\sin \frac{\pi}{2} = f\left(\frac{\pi}{2}\right)$$

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi} \left(-\frac{1}{3} + \frac{1}{15} - \frac{1}{35} + \frac{1}{63} - \frac{1}{99} + \frac{1}{143} \dots \right)$$

$$\left(1 - \frac{1}{\pi} - \frac{1}{2}\right) * \frac{\pi}{2} = \frac{1}{1*3} - \frac{1}{3*5} + \frac{1}{5*7} - \frac{1}{7*9} + \frac{1}{9*11} - \frac{1}{11*13} + \dots$$

$$f(t) = \frac{1}{\pi} + \frac{\sin t}{2} - \frac{2}{\pi} \left[\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \dots \right]$$

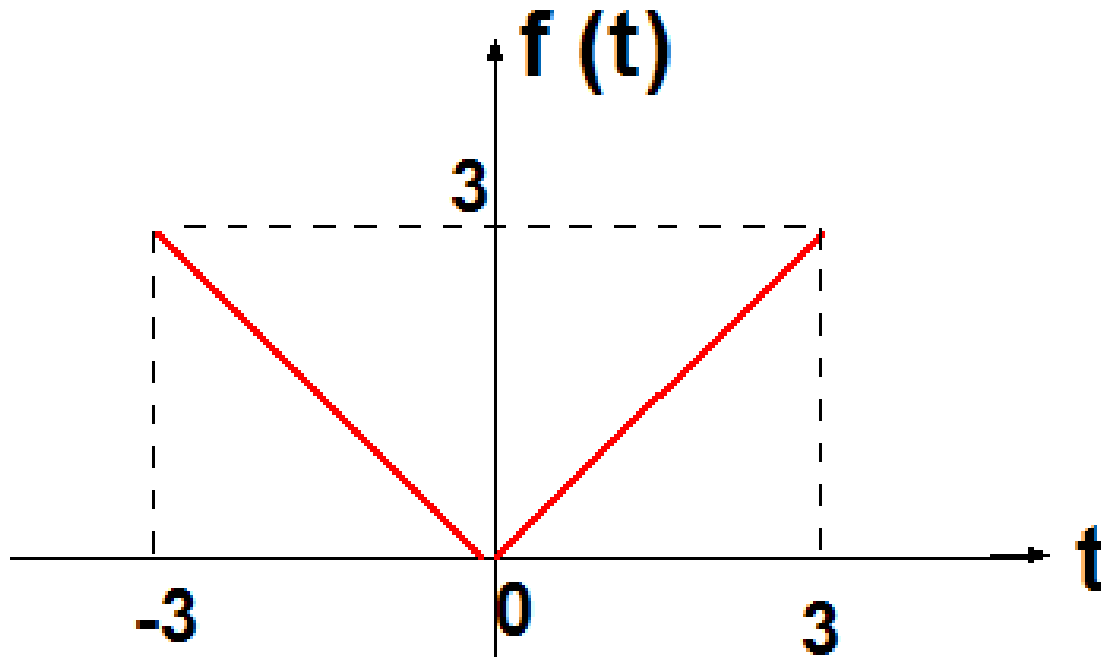


Example :

$$f(x) = \begin{cases} -t & -3 < t < 0 \\ t & 0 < t < 3 \end{cases}$$

Solution :

$$2p = 6, \quad p = 3, \quad \text{take } d = -3$$



$$a_o = \frac{1}{p} \int_d^{d+2p} f(t) dt$$

$$= \frac{1}{3} \left[\int_{-3}^0 -t dt + \int_0^3 t dt \right] = \frac{1}{3} \left(-\frac{t^2}{2} \Big|_{-3}^0 + \frac{t^2}{2} \Big|_0^3 \right) = 3$$

OR:

$$\frac{1}{3}(\text{area}) = \frac{1}{3} [2 * (0.5 * 3 * 3)] = 3$$

$$\begin{aligned}
 a_n &= \frac{1}{p} \int_d^{d+2p} f(t) \cos \frac{n\pi t}{p} dt \\
 &= \frac{1}{3} \left[\int_{-3}^0 -t \cos \frac{n\pi t}{3} dt + \int_0^3 t \cos \frac{n\pi t}{3} dt \right] \\
 &= \frac{1}{3} \left[-\frac{3}{n\pi} t \sin \frac{n\pi t}{3} - \frac{9}{n^2 \pi^2} \cos \frac{n\pi t}{3} \right]_{-3}^0 + \\
 &\quad \frac{1}{3} \left[\frac{3}{n\pi} t \sin \frac{n\pi t}{3} + \frac{9}{n^2 \pi^2} \cos \frac{n\pi t}{3} \right]_0^3 \\
 &= -\frac{1}{3} \left[0 + \frac{9}{n^2 \pi^2} - \left(0 + \frac{9}{n^2 \pi^2} \cos(-\pi n) \right) \right] + \\
 &\quad \frac{1}{3} \left[0 + \frac{9}{n^2 \pi^2} \cos n\pi - \left(0 + \frac{9}{n^2 \pi^2} \cos 0 \right) \right] \\
 &= -\frac{3}{n^2 \pi^2} (1 - \cos n\pi) + \frac{3}{n^2 \pi^2} (\cos n\pi - 1) \quad n \neq 0 \\
 \therefore a_n &= \begin{cases} \frac{6}{n^2 \pi^2} (\cos n\pi - 1) = \frac{-12}{n^2 \pi^2} & (\text{for } n = \text{odd}) \\ 0 & (\text{for } n = \text{even}) \end{cases}
 \end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{p} \int_d^{d+2p} f(t) \sin \frac{n\pi t}{p} dt \\
&= \frac{1}{3} \left[\int_{-3}^0 -t \sin \frac{n\pi t}{3} dt + \int_0^3 t \sin \frac{n\pi t}{3} dt \right] \\
&= -\frac{1}{3} \left[\frac{-3}{n\pi} t \cos \frac{n\pi t}{3} + \frac{9}{n^2 \pi^2} \sin \frac{n\pi t}{3} \right]_{-3}^0 + \\
&\quad \frac{1}{3} \left[-\frac{3}{n\pi} t \cos \frac{n\pi t}{3} + \frac{9}{n^2 \pi^2} \sin \frac{n\pi t}{3} \right]_0^3 \\
&= \frac{3}{n\pi} \cos(-n\pi) - \frac{3}{n\pi} \cos n\pi \\
\therefore b_n &= 0
\end{aligned}$$

$$a_o = 3$$

$$a_n = \frac{-12}{\pi^2 n^2} (n = \text{odd})$$

$$a_n = 0 (n = \text{even})$$

$$b_n = 0$$

$$f(t) = \frac{a_o}{0} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p} \right]$$

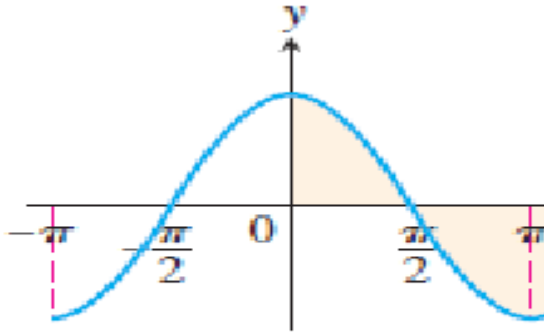
$$\therefore f(t) = \frac{3}{2} - \frac{12}{\pi^2} \left(\frac{1}{1} \cos \frac{\pi t}{3} + \frac{1}{9} \cos \frac{3\pi t}{3} + \frac{1}{25} \cos \frac{5\pi t}{3} + \dots \right)$$

Note :

$f(t)$ is even $\therefore b_n = 0$

$f(t)$ is odd $\therefore a_n = 0$

Even and Odd Functions:



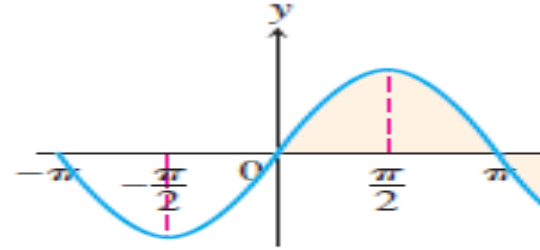
Even

Symmetrical with y-axis

$$g(-x) = g(x)$$

$$\int_{-p}^p g(x) dx = 2 \int_0^p g(x) dx$$

$\cos \frac{n\pi x}{p}$ is even



Odd

Symmetrical with Origin

$$g(-x) = -g(x)$$

$$\int_{-p}^p g(x) dx = 0$$

$\sin \frac{n\pi x}{p}$ is odd

Theorem (1):

$$\text{If } f(x) \text{ is even, } a_n = \frac{2}{p} \int_0^p f(t) \cos \frac{n\pi t}{p} dt$$

$$\text{and } b_n = 0$$

Theorem (2):

$$\text{If } f(x) \text{ is odd, } a_n = 0$$

$$\text{and } b_n = \frac{2}{p} \int_0^p f(t) \sin \frac{n\pi t}{p} dt$$

Product:

$$\textit{Odd} * \textit{Odd} = \textit{Even}$$

$$\textit{Even} * \textit{Even} = \textit{Even}$$

$$\textit{Even} * \textit{Odd} = \textit{Odd}$$

Half – range expansion:

Ex : Find the half – range of :

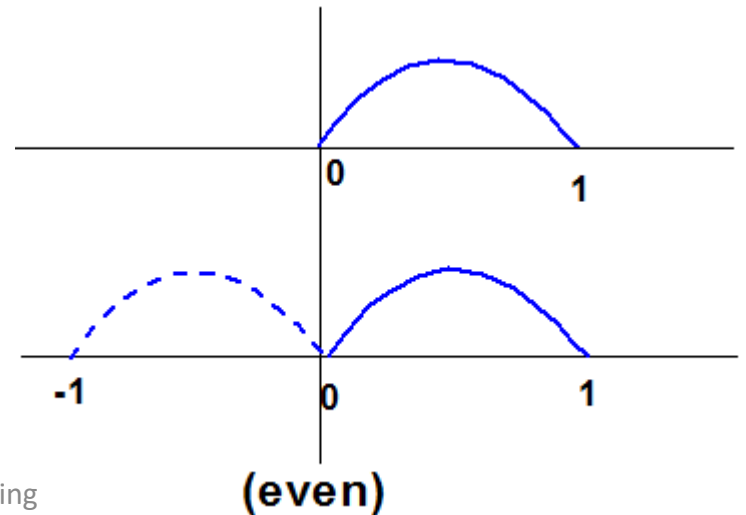
$$f(t) = t - t^2 \quad 0 < t < 1$$

Solution:

(1) *Extend to $(-1,0)$ by
reflection in the y – axis*

$$b_n = 0 \quad (\text{even})$$

$$a_n = \frac{2}{p} \int_0^p f(t) \cos \frac{n \pi t}{p} dt$$



$$\begin{aligned}
 a_n &= \frac{2}{1} \int_0^1 (t - t^2) \cos \frac{n\pi t}{1} dt \\
 &= 2 \left[\left(\frac{t}{n\pi} \sin n\pi t + \frac{1}{n^2 \pi^2} \cos n\pi t \right) - \right. \\
 &\quad \left. \left(\frac{t^2}{n\pi} \sin n\pi t + \frac{2t}{n^2 \pi^2} \cos n\pi t - \frac{2}{n^3 \pi^3} \sin n\pi t \right) \right]_0^1 \\
 &= 2 \left(\frac{\cos n\pi - 1}{n^2 \pi^2} - \frac{2 \cos n\pi}{n^2 \pi^2} \right)
 \end{aligned}$$

$$a_n = 2 \left(\frac{\cos n\pi - 1}{n^2 \pi^2} - \frac{2 \cos n\pi}{n^2 \pi^2} \right)$$

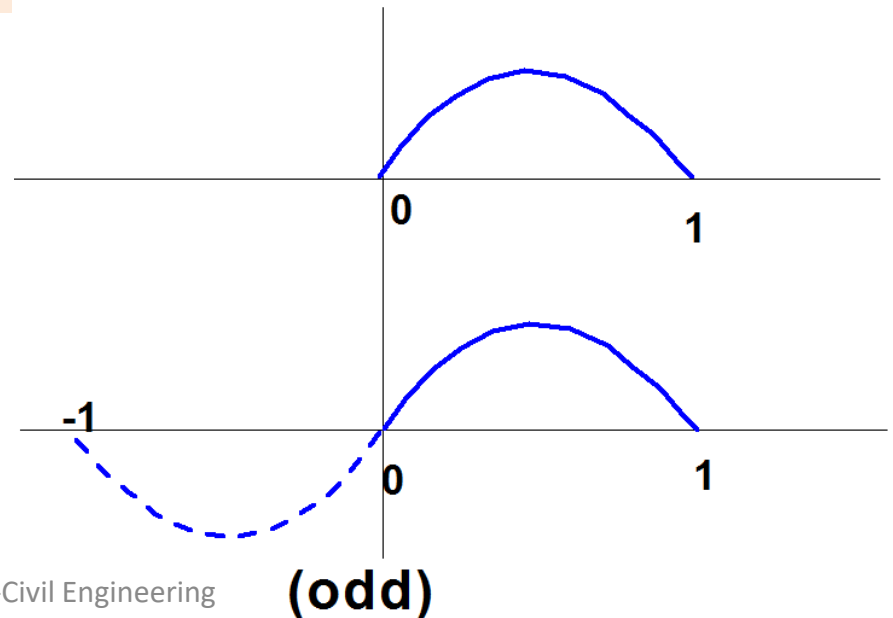
$$\therefore a_n = \frac{-2(1 + \cos n\pi)}{n^2 \pi^2} \quad n \neq 0 \quad (a_n = 0 \quad n = \text{odd})$$

$$a_o = \frac{2}{1} \int_0^1 (t - t^2) dt = 2 \left[\frac{t^2}{2} - \frac{t^3}{3} \right]_0^1 = \frac{1}{3}$$

$$\therefore f(t) = \frac{1}{6} - \frac{4}{\pi^2} \left[\frac{\cos 2\pi t}{4} + \frac{\cos 4\pi t}{16} + \frac{\cos 6\pi t}{36} + \dots \right]$$

(2) *Extend to $(-1,0)$ by reflection in the origin*

$$a_n = 0 \quad (\text{odd})$$
$$b_n = \frac{2}{p} \int_0^p f(t) \sin \frac{n\pi t}{p} dt$$



$$\begin{aligned}
 b_n &= \frac{2}{1} \int_0^1 (t - t^2) \sin \frac{n\pi t}{1} dt \\
 &= 2 \left[\left(\frac{-t}{n\pi} \cos n\pi t + \frac{1}{n^2 \pi^2} \sin n\pi t \right) - \right. \\
 &\quad \left. \left(\frac{-t^2}{n\pi} \cos n\pi t + \frac{2t}{n^2 \pi^2} \sin n\pi t + \frac{2}{n^3 \pi^3} \cos n\pi t \right) \right]_0^1 \\
 &= 2 \left[\left(\frac{-\cos n\pi}{n\pi} \right) - \left(\frac{-\cos n\pi}{n\pi} + \frac{2(\cos n\pi - 1)}{n^3 \pi^3} \right) \right]
 \end{aligned}$$

$$b_n = 2\left[\left(\frac{-\cos n\pi}{n\pi}\right) - \left(\frac{-\cos n\pi}{n\pi} + \frac{2(\cos n\pi - 1)}{n^3\pi^3}\right)\right]$$

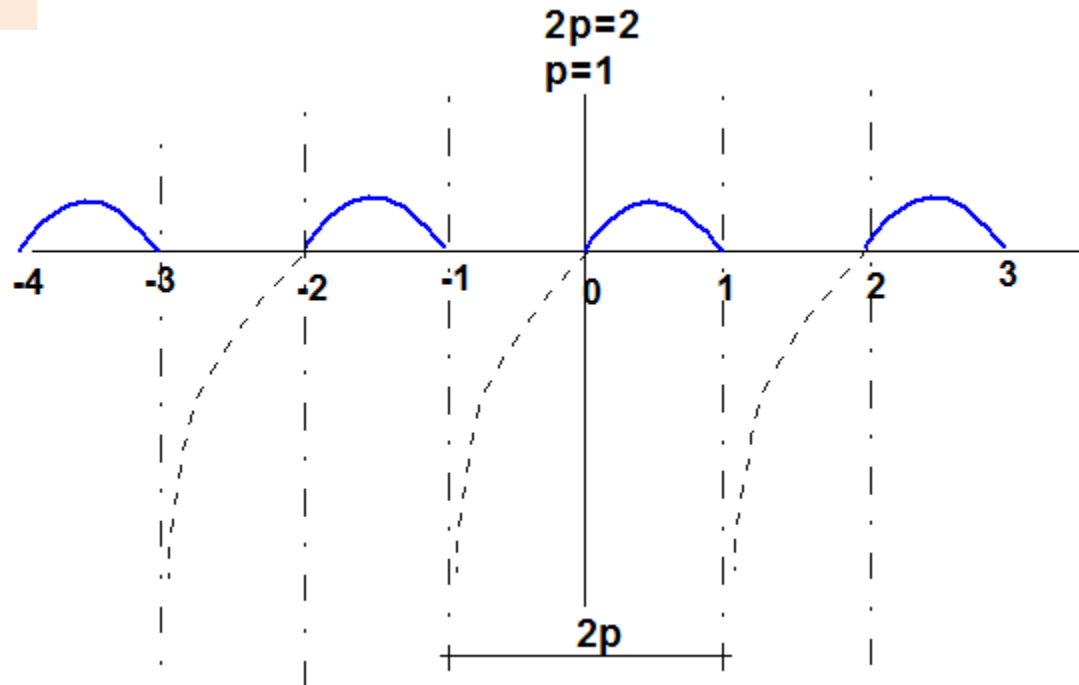
$$\therefore b_n = \frac{4(1 - \cos n\pi)}{n^3\pi^3} \quad (b_n = 0, n = \text{even})$$

$$f(t) = \frac{8}{\pi^3} \left[\frac{\sin \pi t}{1} + \frac{\sin 3\pi t}{27} + \frac{\sin 5\pi t}{125} + \frac{\sin 7\pi t}{343} + \dots \right]$$

(3) Extension of $(t - t^2)$ for $-1 < t < 0$

$$a_n = \frac{1}{p} \int_d^{d+2p} f(t) \cos \frac{n\pi t}{p} dt$$

$$b_n = \frac{1}{p} \int_d^{d+2p} f(t) \sin \frac{n\pi t}{p} dt$$



$$\begin{aligned}
 a_n &= \frac{1}{1} \int_{-1}^1 (t - t^2) \cos \frac{n\pi t}{1} dt \\
 &= \left[\left(\frac{t}{n\pi} \sin n\pi t + \frac{1}{n^2 \pi^2} \cos n\pi t \right) - \right. \\
 &\quad \left. \left(\frac{t^2}{n\pi} \sin n\pi t + \frac{2t}{n^2 \pi^2} \cos n\pi t - \frac{2}{n^3 \pi^3} \sin n\pi t \right) \right]_{-1}^1 \\
 &= \left(\frac{\cos n\pi t}{n^2 \pi^2} - \frac{2 \cos n\pi t}{n^2 \pi^2} \right)_{-1}^1
 \end{aligned}$$

$$\begin{aligned}
 a_n &= \left(\frac{\cos n\pi t}{n^2 \pi^2} - \frac{2 \cos n\pi t}{n^2 \pi^2} \right)_{-1}^1 \\
 &= \frac{1}{n^2 \pi^2} [(\cos n\pi - 2 \cos n\pi) - \\
 &\quad (\cos(-n\pi) - 2 \cos(-n\pi))]
 \end{aligned}$$

Note $[\cos(-n\pi) = \cos n\pi]$

$$\therefore a_n = \frac{-4 \cos n\pi}{n^2 \pi^2} \quad (n \neq 0)$$

$$a_o = \frac{1}{1} \int_{-1}^1 (t - t^2) dt = -\frac{2}{3}$$

$$\begin{aligned}
 b_n &= \frac{1}{1} \int_{-1}^1 (t - t^2) \sin \frac{n\pi t}{1} dt \\
 &= \left[\left(\frac{-t}{n\pi} \cos n\pi t + \frac{1}{n^2 \pi^2} \sin n\pi t \right) - \right. \\
 &\quad \left. \left(\frac{-t^2}{n\pi} \cos n\pi t + \frac{2t}{n^2 \pi^2} \sin n\pi t + \frac{2}{n^3 \pi^3} \cos n\pi t \right) \right]_{-1}^1 \\
 &= \left[\left(\frac{-t}{n\pi} \cos n\pi t \right) - \left(\frac{-t^2}{n\pi} \cos n\pi t + \frac{2}{n^3 \pi^3} \cos n\pi t \right) \right]_{-1}^1
 \end{aligned}$$

$$b_n = \left[\left(\frac{-\cos n\pi}{n\pi} + \frac{\cos n\pi}{n\pi} - \frac{2}{n^3\pi^3} \cos n\pi \right) - \left(\frac{\cos(-n\pi)}{n\pi} + \frac{\cos(-n\pi)}{n\pi} - \frac{2}{n^3\pi^3} \cos(-n\pi) \right) \right]$$

$$\therefore b_n = -2 \frac{\cos n\pi}{n\pi}$$

$$\therefore a_n = \frac{-4 \cos n\pi}{n^2 \pi^2}$$

$$a_o = -\frac{2}{3}$$

$$b_n = \frac{-2 \cos n\pi}{n\pi}$$

$$\therefore f(t) = -\frac{1}{3} + \frac{4}{\pi^4} \left(\frac{\cos \pi t}{1} - \frac{\cos 2\pi t}{4} + \frac{\cos 3\pi t}{9} - \frac{\cos 4\pi t}{16} + \frac{\cos 5\pi t}{25} \dots \right) + \frac{2}{\pi} \left(\frac{\sin \pi t}{1} - \frac{\sin 2\pi t}{2} + \frac{\sin 3\pi t}{3} - \frac{\sin 4\pi t}{4} \dots \right)$$

Alternative forms of fourier Series:

The complex exponetial form is obtained:

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{\frac{ni\pi t}{p}} + e^{-\frac{ni\pi t}{p}}}{2} + b_n \frac{e^{\frac{ni\pi t}{p}} - e^{-\frac{ni\pi t}{p}}}{2i} \right)$$

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} e^{\frac{ni\pi t}{p}} + \frac{a_n + ib_n}{2} e^{-\frac{ni\pi t}{p}} \right)$$

Define:

$$C_o = \frac{a_o}{2}; \quad C_n = \frac{a_n - ib_n}{2}; \quad C_{-n} = \frac{a_n + ib_n}{2}$$

$$\therefore f(t) = \sum_{-\infty}^{\infty} C_n e^{\frac{ni\pi t}{p}}$$

$$\left\langle f(t) = \sum_{-\infty}^{\infty} C_n e^{\frac{ni\pi t}{p}} \right\rangle$$

$$C_o = \frac{1}{2} a_o = \frac{1}{2p} \int_d^{d+2p} f(t) dt$$

$$\begin{aligned} C_n &= \frac{a_n - ib_n}{2} \\ &= \frac{1}{2} \left[\frac{1}{p} \int_d^{d+2p} f(t) \cos \frac{n\pi t}{p} dt - i \frac{1}{p} \int_d^{d+2p} f(t) \sin \frac{n\pi t}{p} dt \right] \\ &= \frac{1}{2p} \int_d^{d+2p} f(t) \left(\cos \frac{n\pi t}{p} - i \sin \frac{n\pi t}{p} \right) dt \end{aligned}$$

$$\therefore C_n = \frac{1}{2p} \int_d^{d+2p} f(t) e^{\frac{-ni\pi t}{p}} dt$$

$$C_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2p} \int_d^{d+2p} f(t) e^{\frac{ni\pi t}{p}} dt$$

$$\left\langle f(t) = \sum_{-\infty}^{\infty} C_n e^{\frac{ni\pi t}{p}} \right\rangle$$

$$C_o = \frac{1}{2} a_o = \frac{1}{2p} \int_d^{d+2p} f(t) dt$$

$$C_n = \frac{1}{2p} \int_d^{d+2p} f(t) e^{\frac{-ni\pi t}{p}} dt$$

$$C_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2p} \int_d^{d+2p} f(t) e^{\frac{ni\pi t}{p}} dt$$

OR:

$$\left\langle C_n = \frac{1}{2p} \int_d^{d+2p} f(t) e^{\frac{-ni\pi t}{p}} dt \right\rangle \quad (n = +ve, -ve \text{ or } 0)$$

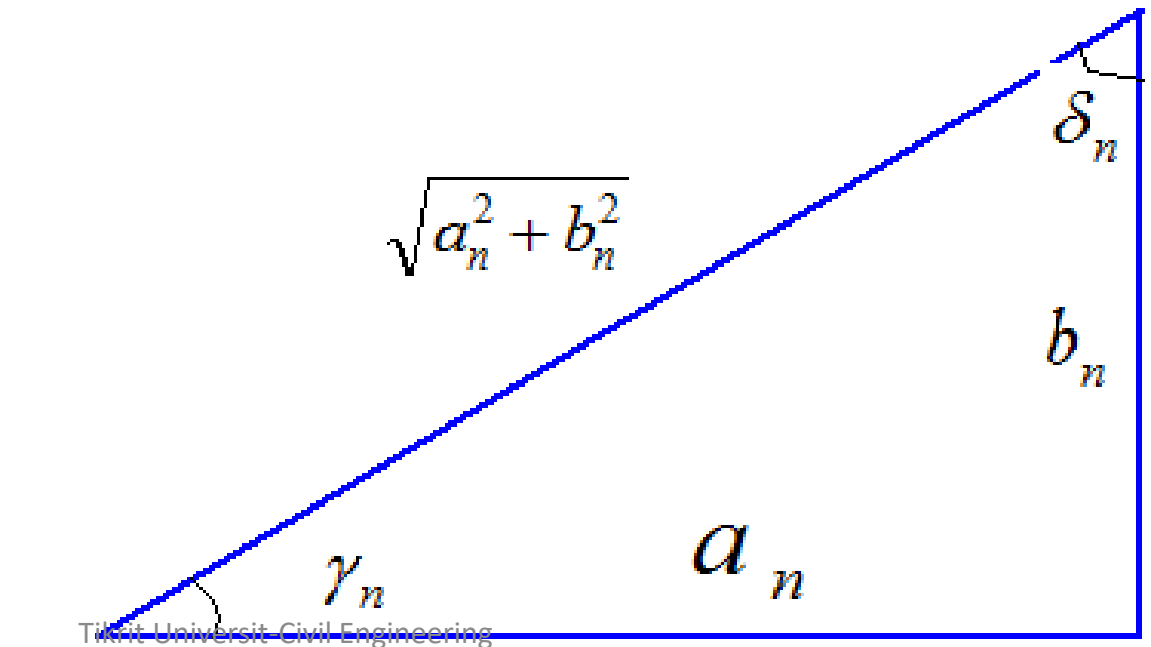
Another trigonometric forms of fourier Series:

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p} \right)$$

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos \frac{n\pi t}{p} + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin \frac{n\pi t}{p} \right)$$

$$\text{Let } A_o = \frac{a_o}{2}$$

$$A_n = \sqrt{a_n^2 + b_n^2}$$



$$\begin{aligned}
 f(t) &= A_o + \sum_{n=1}^{\infty} A_n \left(\cos \frac{n\pi t}{p} \cos \gamma_n + \sin \frac{n\pi t}{p} \sin \gamma_n \right) \\
 &= A_o + \sum_{n=1}^{\infty} A_n \cos \left(\frac{n\pi t}{p} - \gamma_n \right)
 \end{aligned}$$

OR:

$$\begin{aligned}
 f(t) &= A_o + \sum_{n=1}^{\infty} A_n \left(\cos \frac{n\pi t}{p} \sin \delta_n + \sin \frac{n\pi t}{p} \cos \delta_n \right) \\
 &= A_o + \sum_{n=1}^{\infty} A_n \sin \left(\frac{n\pi t}{p} + \delta_n \right)
 \end{aligned}$$

Note:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

By solving the above equations:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Example: Find the complex form of Fourier

Series of: $f(t) = e^{-t} \quad -1 < t < 1$

Solution:

$$C_n = \frac{1}{2p} \int_d^{d+2p} f(t) e^{-\frac{ni\pi t}{p}} dt = \frac{1}{2} \int_{-1}^1 e^{-t} e^{-ni\pi t} dt$$

$$= \frac{1}{2} \int_{-1}^1 e^{-(1+ni\pi)t} dt = \frac{1}{2} \left[\frac{e^{-(1+ni\pi)t}}{-(1+ni\pi)} \right]_{-1}^1$$

$$\begin{aligned}
 C_n &= \frac{1}{2} \left[\frac{e^{-(1+ni\pi)t}}{-(1+ni\pi)} \right]_{-1}^1 \\
 &= \frac{e^{-(1+ni\pi)} - e^{(1+ni\pi)}}{-2(1+ni\pi)} = \frac{e \cdot e^{ni\pi} - e^{-1} \cdot e^{-ni\pi}}{2(1+ni\pi)}
 \end{aligned}$$

$$e^{i\pi} = \cos \pi + i \sin \pi = -1$$

$$e^{-i\pi} = \cos \pi - i \sin \pi = -1$$

$$e^{ni\pi} = e^{-ni\pi} = (-1)^n$$

$$C_n = \frac{e \cdot e^{ni\pi} - e^{-1} \cdot e^{-ni\pi}}{2(1 + ni\pi)} = \frac{(-1)^n}{1 + ni\pi} \frac{e - e^{-1}}{2}$$

$$= \frac{(-1)^n (1 - ni\pi) \sinh 1}{1 + n^2 \pi^2}$$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{ni\pi t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - ni\pi) \sinh 1}{1 + n^2 \pi^2} e^{ni\pi t}$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \& \quad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

This can be converted into trigonometric series:

$$C_n = \frac{a_n - i b_n}{2} \quad \& \quad C_{-n} = \frac{a_n + i b_n}{2}$$

Solve the above Eqs.

$$a_n = C_n + C_{-n} \quad \& \quad b_n = i(C_n - C_{-n})$$

$$\begin{aligned} a_n &= \frac{(-1)^n (1 - n i \pi) \sinh 1}{1 + n^2 \pi^2} + \frac{(-1)^{-n} (1 + n i \pi) \sinh 1}{1 + n^2 \pi^2} \\ &= \frac{(-1)^n (2 \sinh 1)}{1 + n^2 \pi^2} \\ b_n &= i \left[\frac{(-1)^n (1 - n i \pi) \sinh 1}{1 + n^2 \pi^2} - \frac{(-1)^{-n} (1 + n i \pi) \sinh 1}{1 + n^2 \pi^2} \right] \\ &= \frac{(-1)^n 2 n \pi \sinh 1}{1 + n^2 \pi^2} \end{aligned}$$

$$a_n = \frac{(-1)^n (2 \sinh 1)}{1 + n^2 \pi^2}$$

$$b_n = \frac{(-1)^n 2n \pi \sinh 1}{1 + n^2 \pi^2}$$

$$\begin{aligned} \therefore f(t) = & \sinh 1 - 2 \sinh 1 \left(\frac{\cos \pi t}{1 + \pi^2} - \frac{\cos 2\pi t}{1 + 4\pi^2} + \frac{\cos 3\pi t}{1 + 9\pi^2} \right. \\ & \left. - \frac{\cos 4\pi t}{1 + 16\pi^2} + \dots \right) - 2\pi \sinh 1 \left(\frac{\sin \pi t}{1 + \pi^2} - \frac{2\sin 2\pi t}{1 + 4\pi^2} + \right. \\ & \left. \frac{3\sin 3\pi t}{1 + 9\pi^2} - \frac{4\sin 4\pi t}{1 + 16\pi^2} + \dots \right) \end{aligned}$$

Note: $(-1)^0 = 1$ & $(-1)^n = (-1)^{-n}$

Sum of functions:

Fourier coefficients of Sum.

$$f_1 + f_2$$

are Sum. of f_1 & f_2

Example:

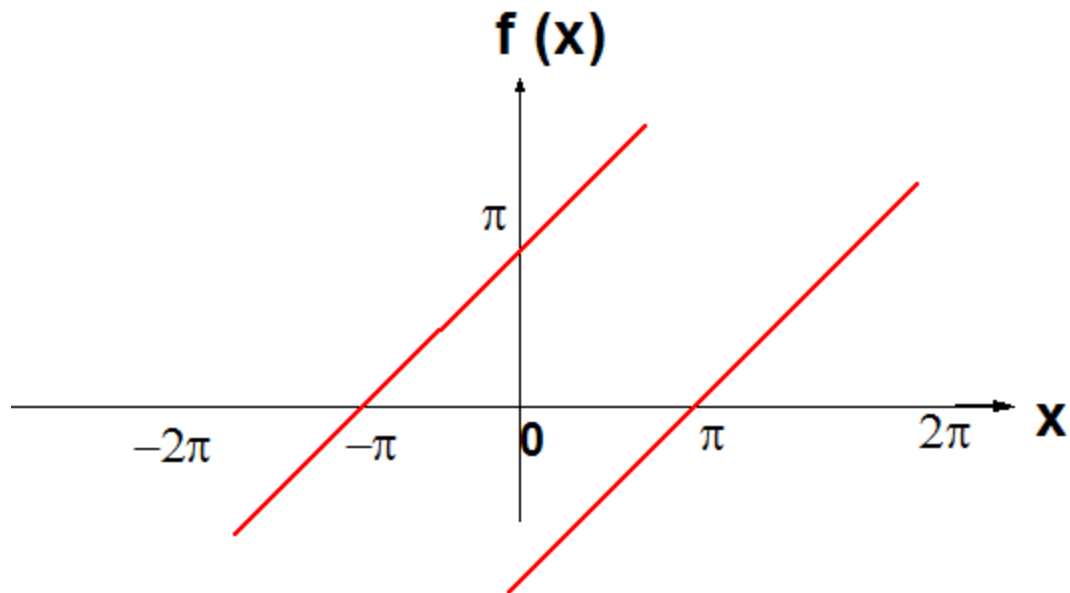
$$f(x) = x + \pi$$

$$-\pi < x < \pi$$

$$f = f_1 + f_2 \quad \text{where} \quad f_1 = x \quad \& \quad f_2 = \pi$$

Solution:

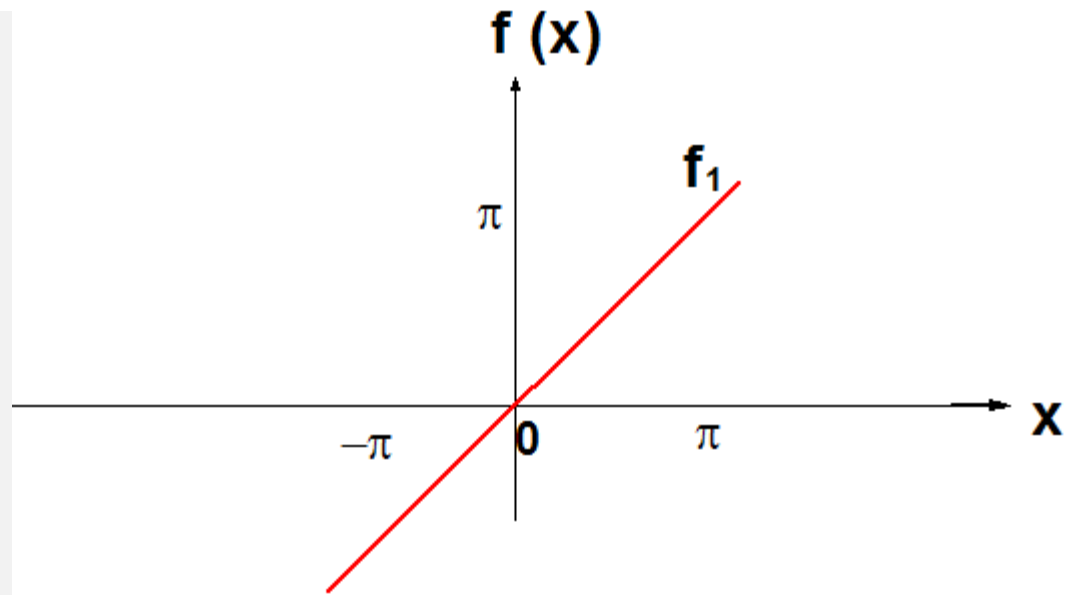
The coefficients of Fourier series of f_2 are zero except for $a_0 = 2\pi$



Since f_1 is odd, $a_n = 0$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f_1(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$



$$b_n = \frac{2}{\pi} \left[-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_0^{\pi}$$
$$= -\frac{2}{\pi} \left[+\frac{\pi}{n} \cos n\pi - 0 \right] = -\frac{2}{n} \cos n\pi$$

$$f_2 \Rightarrow a_0 = 2\pi$$

$$f_1 \Rightarrow a_n = 0 \quad \& \quad b_n = -\frac{2}{n} \cos n\pi$$

$$(b_1 = 2, \quad b_2 = -\frac{2}{2}, \quad b_3 = \frac{2}{3}, \quad b_4 = -\frac{2}{4}, \dots)$$

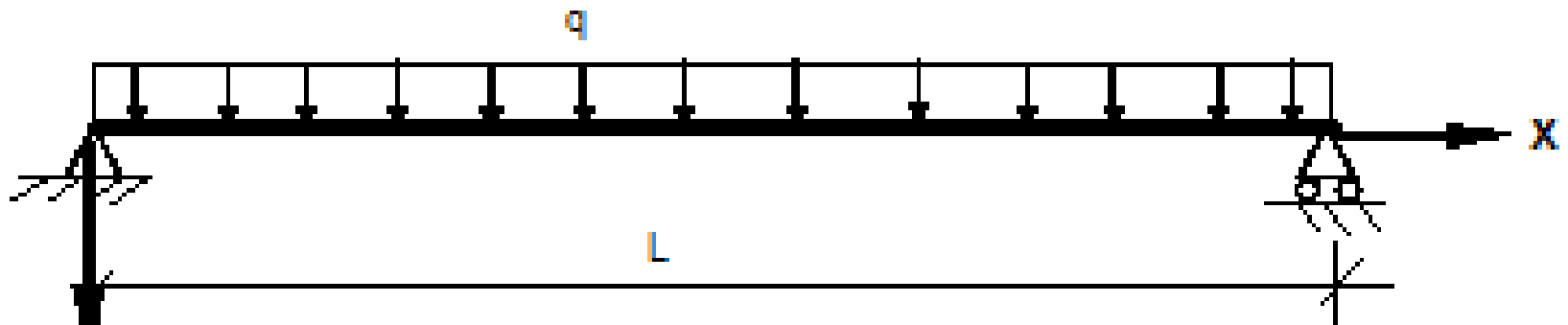
$$\begin{aligned} \therefore f(x) &= \pi + 2\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \dots \\ &= \pi + 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right) \end{aligned}$$

Applications:

* Deflection of simply supported beams:

Note:

$EIy'' = M$ (moment), $EIy''' = V$ (shear) & $EIy^{iv} = W$ (load)



conditions

$$\begin{array}{ll} y(0)=y(L)=0 & \text{No Defl. at ends} \\ y''(0)=y''(L)=0 & \text{No Mom. at ends} \end{array}$$

$$EI \frac{d^4 y}{dx^4} = q(x)$$

$$\frac{d^4 y}{dx^4} = \frac{q(x)}{EI} \quad (1)$$

y = y(x), to satisfy the boundary conditions at the ends; x = 0 & x = L

$$\text{Try } y = \sum_{m=1}^{\infty} C_m \sin \frac{m \pi x}{L} \quad (2)$$

$$y = 0, \text{ where } \sin 0 = \sin m \pi = 0$$

Expand the load by $\frac{1}{2}$ range (sine series)

$$q(x) = \sum b_m \sin \frac{m \pi x}{L} \quad (3)$$

$$b_m = \frac{2}{L} \int_0^L q(x) \sin \frac{m \pi x}{L} dx$$

Subst. Eqs. (2) & (3) into Eq. (1):

$$\sum C_m \left(\frac{m \pi}{L} \right)^4 \sin \frac{m \pi x}{L} = \frac{1}{EI} \sum b_m \sin \frac{m \pi x}{L}$$

$$C_m \left(\frac{m \pi}{L} \right)^4 = \frac{b_m}{EI}$$

$$\therefore C_m = \frac{b_m}{EI} \left(\frac{L}{m \pi} \right)^4$$

For Uniform Distributed Load :

$$b_m = \frac{2}{L} \int_0^L q \sin \frac{m \pi x}{L} dx$$

$$= -\frac{2q}{L} \frac{L}{m \pi} \cos \frac{m \pi x}{L} \Big|_0^L$$

$$= -\frac{2q}{m \pi} [\cos m \pi - \cos 0]$$

$$= -\frac{2q}{m \pi} [(-1)^m - 1]$$

$$\therefore b_m = \begin{cases} \frac{4q}{m \pi} & m = \text{odd} \\ 0 & m = \text{even} \end{cases}$$

$$\therefore C_m = \frac{4q}{EI m \pi} \left(\frac{L}{m \pi} \right)^4 = \frac{4qL^4}{EI m^5 \pi^5}$$

$$y = \sum C_m \sin \frac{m \pi x}{L}$$

$$= \sum_{m=1,3,5,\dots}^{\infty} \frac{4qL^4}{EI m^5 \pi^5} \sin \frac{m \pi x}{L}$$

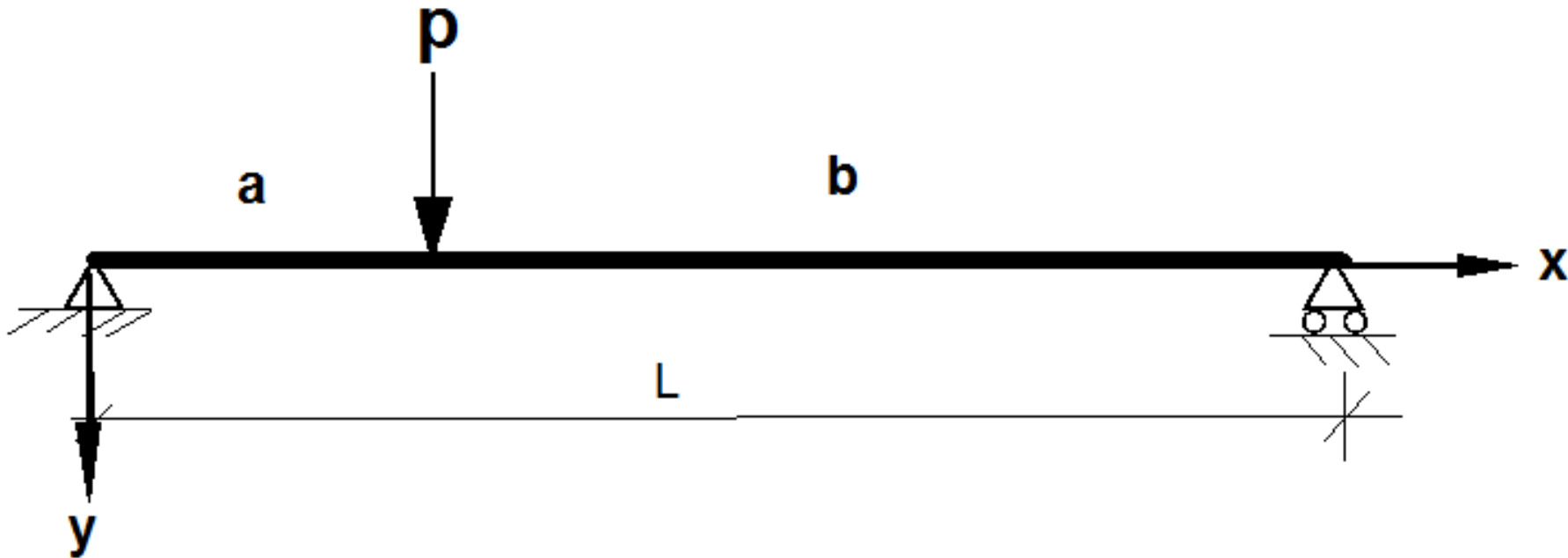
Take $m=1$ at $x = \frac{L}{2}$

$$y = \frac{4qL^4}{EI \pi^5} \sin \frac{\pi L/2}{L} = \frac{4qL^4}{EI \pi^5} \sin \frac{\pi}{2}$$

$$y = 0.013071054 \frac{qL^4}{EI}$$

$$y_{exact} = \frac{5qL^4}{384EI} = 0.01302833 \frac{qL^4}{EI}$$

Example: Simply supported beam,
with concentrated load.



From previous example:

$$\frac{d^4 y}{dx^4} = \frac{q(x)}{EI} \quad (1)$$

$$y = \sum_{m=1}^{\infty} C_m \sin \frac{m \pi x}{L} \quad (2)$$

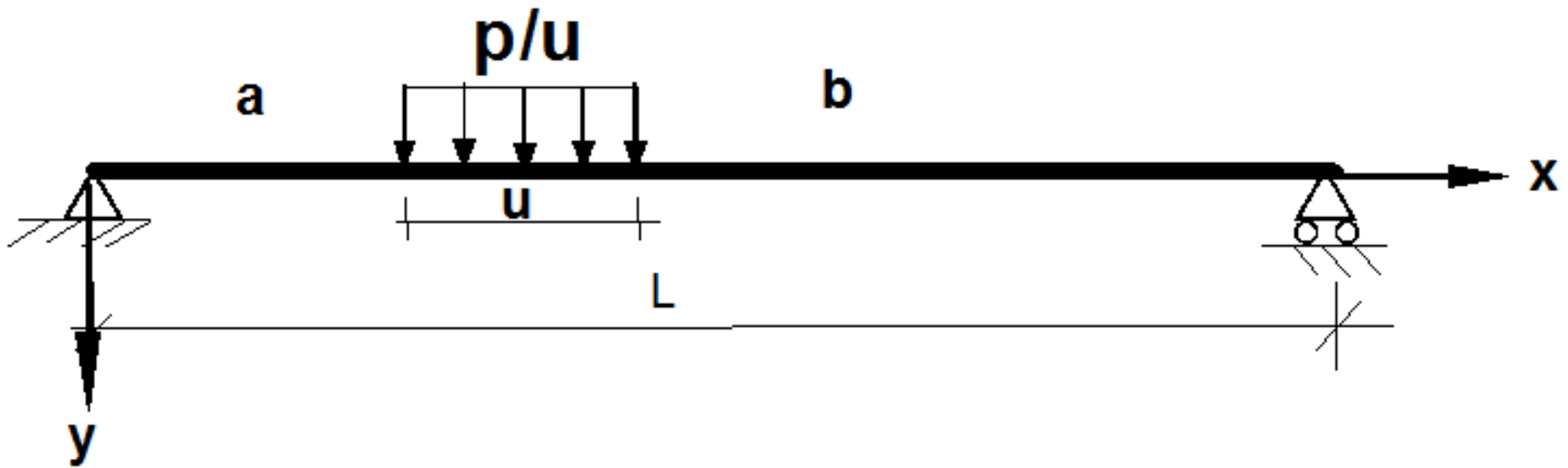
$$q(x) = \sum b_m \sin \frac{m \pi x}{L} \quad (3)$$

$$b_m = \frac{2}{L} \int_0^L q(x) \sin \frac{m \pi x}{L} dx$$

$$b_m = \frac{2}{L} \int_a^a p \sin \frac{m \pi x}{L} dx = 0 \text{ ?????????}$$

Distribute p over small distance u :

$$b_m = \frac{2}{L} \int_a^{a+u} \frac{p}{u} \sin \frac{m\pi x}{L} dx$$



$$b_m = -\frac{2}{L} \frac{p}{u} \frac{L}{m\pi} \cos \frac{m\pi x}{L} \Big|_a^{a+u}$$

$$b_m = -\frac{2p}{um\pi} \left[\cos \frac{m\pi(a+u)}{L} - \cos \frac{m\pi a}{L} \right]$$

$$= -\frac{2p}{um\pi} \left[\cos \frac{m\pi a}{L} \cos \frac{m\pi u}{L} - \right.$$

$$\left. \sin \frac{m\pi u}{L} \sin \frac{m\pi a}{L} - \cos \frac{m\pi a}{L} \right]$$

$$u \rightarrow 0 \quad \therefore \cos \frac{m\pi u}{L} \rightarrow 1$$

$$\& \sin \frac{m\pi u}{L} \rightarrow \frac{m\pi u}{L}$$

$$b_m = -\frac{2p}{um\pi} \left[\cos \frac{m\pi a}{L} - \frac{m\pi u}{L} \sin \frac{m\pi a}{L} - \cos \frac{m\pi a}{L} \right]$$

$$= -\frac{2p}{um\pi} \left[-\frac{m\pi u}{L} \sin \frac{m\pi a}{L} \right]$$

$$\therefore b_m = \frac{2p}{L} \sin \frac{m\pi a}{L}$$

$$C_m \left(\frac{m\pi}{L} \right)^4 = \frac{b_m}{EI} \Rightarrow C_m = \frac{b_m}{EI} \left(\frac{L}{m\pi} \right)^4$$

$$C_m = \frac{2p \sin \frac{m\pi a}{L}}{LEI} \frac{L^4}{m^4 \pi^4}$$

$$\therefore C_m = \frac{2pL^3}{E I m^4 \pi^4} \sin \frac{m\pi a}{L}$$

$$C_m = \frac{2pL^3}{EI\pi^4} \sin \frac{m\pi a}{L}$$

$$y = \sum \frac{2pL^3}{EI\pi^4} \sin \frac{m\pi a}{L} \sin \frac{m\pi x}{L}$$

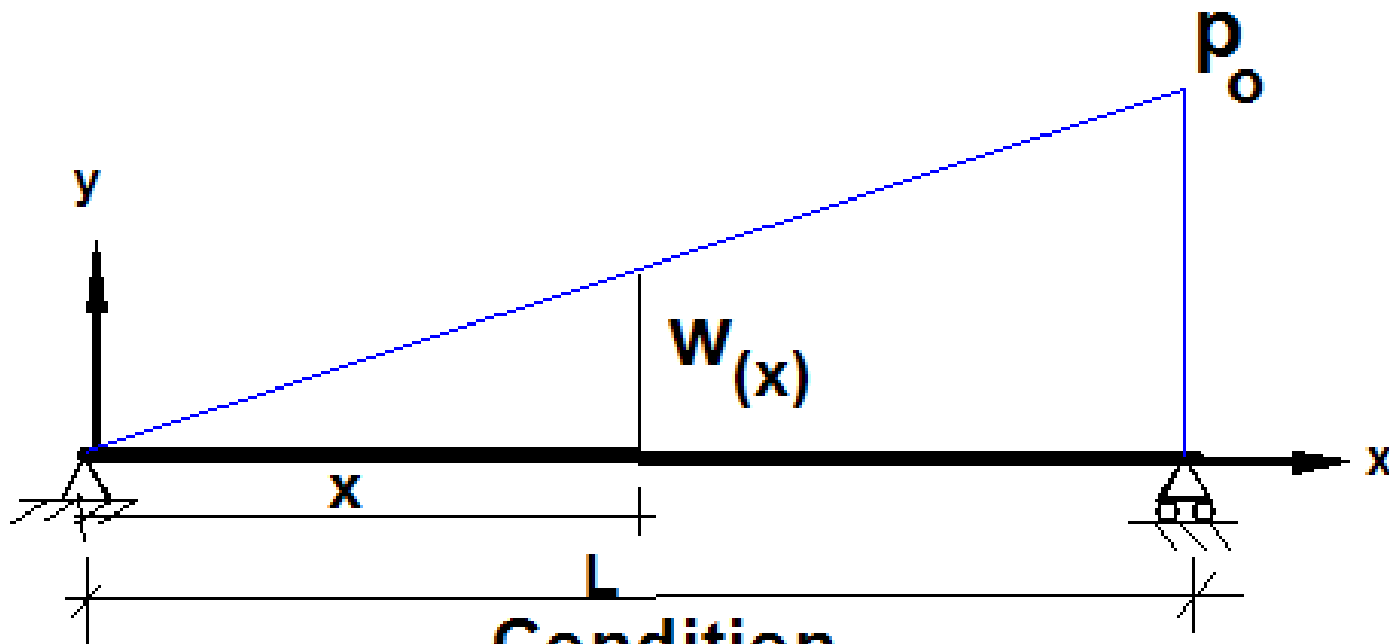
$$\text{For } a = \frac{L}{2}, \quad x = \frac{L}{2} \quad \& \quad m = 1$$

$$y = \frac{2pL^3}{EI\pi^4} \sin \frac{\pi}{2} \sin \frac{\pi}{2}$$

$$\therefore y = \frac{2pL^3}{EI\pi^4} = 0.0205319 \frac{pL^3}{EI}$$

$$y_{exact} = 0.0208333 \frac{pL^3}{EI}$$

Example: Load $w_{(x)} = \frac{p_o}{L} x$ and neglected the Wt. of the beam. Find the deflection curve.



Condition

$$y(0) = y(L) = 0$$
$$y''(0) = y''(L) = 0$$

Solution:

$$EIy'' = M$$

$$EIy''' = V$$

$$EIy^{iv} = -W_{(x)} = -p_o \frac{x}{L} \quad (1)$$

We choose a sine series for y that satisfies Eq.(1)

$$y = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \quad (2)$$

Because each term (and hence the Sum. itself) satisfied the end conditions. Hence expand the load deflection $\left(-p_o \frac{x}{L}\right)$ in a Fourier sine series on interval $(0, L)$:

$$b_n = \frac{2}{L} \int_0^L \left(\frac{-p_o x}{L}\right) \sin \frac{n\pi x}{L} dx$$

$$= \frac{2p_o}{L^2} \left[x \frac{L}{n\pi} \cos \frac{n\pi x}{L} - \frac{L^2}{n^2 \pi^2} \sin \frac{n\pi x}{L} \right]_0^L$$

$$= \frac{2p_o}{L^2} \left[\left(\frac{L^2}{n\pi} \cos \frac{n\pi L}{L} - 0 \right) - \frac{L^2}{n^2 \pi^2} (\sin n\pi - \sin 0) \right]$$

$$\therefore b_n = \frac{2p_o}{L^2} \frac{L^2}{n\pi} \cos n\pi = \frac{2p_o}{n\pi} (-1)^n$$

$$\therefore -p_o \frac{x}{L} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$= \frac{2p_o}{\pi} \sum \frac{(-1)^n}{n} \sin \frac{n\pi x}{L}$$

Now compute y^{iv} from Eq. (2) and put into Eq. (1):

$$EI \frac{\pi^4}{L^4} \sum n^4 C_n \sin \frac{n\pi x}{L} = \frac{2p_o}{\pi} \sum \frac{(-1)^n}{n} \sin \frac{n\pi x}{L}$$

The coefficients of like term must be equal:

$$EI \frac{\pi^4}{L^4} n^4 C_n = \frac{2p_o}{\pi} \frac{(-1)^n}{n}$$

$$\therefore C_n = (-1)^n \frac{2p_o L^4}{EI \pi^5 n^5}$$

$$y = \sum_{n=1}^{\infty} C_n \sin \frac{n \pi x}{L}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{2 p_o L^4}{EI \pi^5 n^5} \sin \frac{n \pi x}{L}$$

$$n=1$$

$$y = - \frac{2 p_o L^4}{EI \pi^5} \sin \frac{\pi x}{L}$$

$$\text{If } x = \frac{L}{2}$$

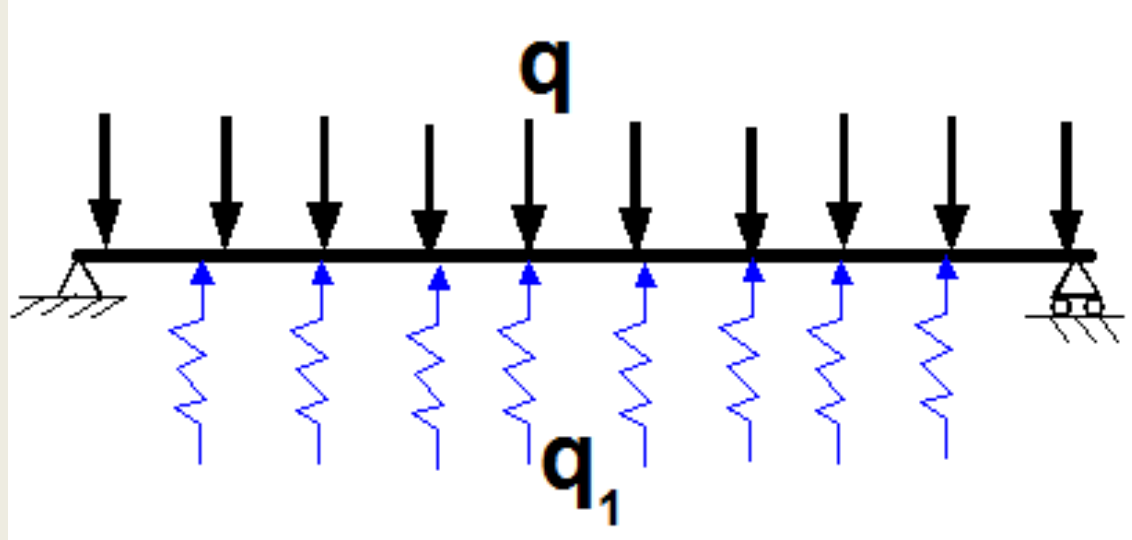
$$y = - \frac{2 p_o L^4}{EI \pi^5} \sin \frac{\pi}{2} = -0.00654 \frac{p_o L^4}{EI}$$

* Beam on elastic foundation:

$$q_1 = k y$$

$$q_{(x)} = q - k y$$

$$\frac{d^4 y}{d x^4} = \frac{q - k y}{EI}$$



$$\frac{d^4 y}{d x^4} + \frac{k y}{EI} = \frac{q}{EI} \quad (1)$$

$$\text{Try } y = \sum C_m \sin \frac{m \pi x}{L}$$

$$q_{(x)} = \sum b_m \sin \frac{m \pi x}{L}$$

Subst. the above into Eq. (1):

$$\sum C_m \left(\frac{m \pi}{L}\right)^4 \sin \frac{m \pi x}{L} + \frac{k}{EI} \sum C_m \sin \frac{m \pi x}{L} = \frac{1}{EI} \sum b_m \sin \frac{m \pi x}{L}$$

$$C_m \left(\frac{m \pi}{L}\right)^4 + \frac{k}{EI} C_m = \frac{b_m}{EI}$$

$$C_m \left[\left(\frac{m \pi}{L}\right)^4 + \frac{k}{EI} \right] = \frac{b_m}{EI}$$

$$\therefore C_m = \frac{b_m}{EI \left[\left(\frac{m \pi}{L}\right)^4 + \frac{k}{EI} \right]}$$

For simply supported beam under uniform dead load on elastic foundation:

$$b_m = \frac{4q}{m\pi} \quad m = 1, 3, 5, \dots$$

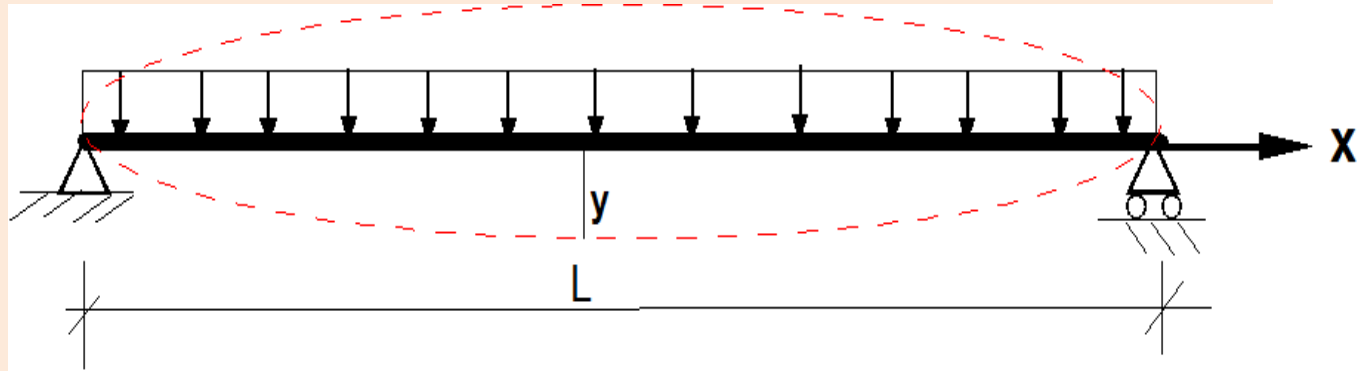
$$\therefore C_m = \frac{4q}{m\pi EI \left[\left(\frac{m\pi}{L} \right)^4 + \frac{k}{EI} \right]}$$

$$y = \sum \frac{4q \sin \frac{m\pi x}{L}}{m\pi EI \left[\left(\frac{m\pi}{L} \right)^4 + \frac{k}{EI} \right]}$$

* Free vibration of simply supported beams:

Free vibration is started
after the load is removed

$$\frac{d^4 y}{dx^4} = \frac{q}{EI}$$



$$\text{Inertia force} = -\frac{\rho A}{EI} \frac{\partial^2 y}{\partial t^2} \left(\begin{array}{l} \rho = \text{Wt. per unit volume} \\ A = \text{cross section area} \end{array} \right)$$

$$\frac{d^4 y}{dx^4} + \frac{\rho A}{EI} \frac{\partial^2 y}{\partial t^2} = 0 \quad (1)$$

$$\text{Try } y(x,t) = \sum_{n=1}^{\infty} f(t) \sin \frac{n \pi x}{L} \quad (2)$$

Subst. Eq.(2) into Eq. (1):

$$\sum_{n=1}^{\infty} \left(\frac{n \pi}{L}\right)^4 f(t) \sin \frac{n \pi x}{L} + \frac{\rho A}{EI} \sum f''(t) \sin \frac{n \pi x}{L} = 0$$

$$\left(\frac{n \pi}{L}\right)^4 f(t) + \frac{\rho A}{EI} f''(t) = 0$$

OR:

$$\frac{\rho A}{EI} f''(t) + \left(\frac{n \pi}{L}\right)^4 f(t) = 0$$

$$f''(t) + \frac{EI}{\rho A} \left(\frac{n \pi}{L}\right)^4 f(t) = 0$$

$$f''(t) + \frac{EI}{\rho A} \left(\frac{n\pi}{L} \right)^4 f(t) = 0$$

$$\text{Let } \frac{EI}{\rho A} \left(\frac{n\pi}{L} \right)^4 = \alpha^2$$

$$\therefore f''(t) + \alpha^2 f(t) = 0$$

$$\text{Let } f(t) = c e^{mt}$$

$$\therefore m^2 + \alpha^2 = 0 \Rightarrow m = \pm \alpha i$$

$$f(t) = c_1 \cos \alpha t + c_2 \sin \alpha t$$

$$\therefore y(x, t) = \sum (c_1 \cos \alpha t + c_2 \sin \alpha t) \sin \frac{n\pi x}{L}$$

$$\text{At } t = 0 \Rightarrow \text{velocity} = 0 = \frac{\partial y}{\partial t}$$

$$\frac{\partial y}{\partial t} = \sum (-c_1 \alpha \sin \alpha t + c_2 \alpha \cos \alpha t) \sin \frac{n \pi x}{L}$$

$$\frac{\partial y}{\partial t} = 0 \text{ @ } t = 0$$

$$0 = \sum (-c_1 \alpha \sin 0 + c_2 \alpha \cos 0) \sin \frac{n \pi x}{L}$$

$$0 = \sum (0 + c_2 \alpha) \Rightarrow c_2 = 0$$

$$y = \sum c_1 \cos \alpha t \sin \frac{n \pi x}{L}$$

At $t = 0$, $y(x,0) = y$ (static)

For uniform distribution load:

$$y = \sum \frac{4q L^4}{EI n^5 \pi^5} \sin \frac{n \pi x}{L}$$

At $t = 0$, $y(x,0) = y$ (static):

$$\sum \frac{4q L^4}{EI n^5 \pi^5} \sin \frac{n \pi x}{L} = \sum c_1 \cos 0 \sin \frac{n \pi x}{L}$$

$$\therefore c_1 = \frac{4qL^4}{EI n^5 \pi^5}$$

$$\therefore y(x,t) = \sum \frac{4qL^4}{EI n^5 \pi^5} \cos \alpha t \sin \frac{n \pi x}{L}$$

The Fourier Integral:

Many problems do not involve periodic function, and an non periodic functions cannot be handled directly by Fourier series. If in a periodic function:, $f_p(t)$, we let approaches infinity, then $f(t)$ is no longer periodic.

Begin with complex form:

$$f_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{ni\pi t}{p}}$$

$$C_n = \frac{1}{2p} \int_{-p}^p f_p(t) e^{-\frac{ni\pi t}{p}} dt$$

$$= \frac{1}{2p} \int_{-p}^p f_p(\tau) e^{-\frac{ni\pi \tau}{p}} d\tau$$

$$\therefore f_p(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2p} \int_{-p}^p f_p(\tau) e^{-\frac{ni\pi \tau}{p}} d\tau \right] e^{\frac{ni\pi t}{p}}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-p}^p f_p(\tau) e^{-\frac{ni\pi \tau}{p}} d\tau \right] e^{\frac{ni\pi t}{p}} \frac{\pi}{p}$$

Let frequency $w_n = \frac{n\pi}{p}$ and $\Delta w = \frac{\pi}{p}$

$$f_p(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} e^{i w_n t} \int_{-p}^p f_p(\tau) e^{-i w_n \tau} d\tau \right] \Delta w$$

$$\text{Let } F(w) = \frac{e^{i w_n t}}{2\pi} \int_{-p}^p f_p(\tau) e^{-i w_n \tau} d\tau$$

$$\therefore f_p(t) = \sum_{n=-\infty}^{\infty} F(w_n) \Delta w \quad (1)$$

The limit of a sum of the form(1) is the

integral, $\int_{-\infty}^{\infty} F(w) dw$

Since $p \rightarrow \infty$ implies $\Delta w \rightarrow 0$, the non periodic limit of $f_p(t)$, say $f(t)$ can be written as the integral;

$$f(t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} e^{iwt} \int_{-\infty}^{\infty} f(\tau) e^{-i w \tau} d\tau \right] dw \quad (2)$$

The other from:

$$f(t) = \int_{-\infty}^{\infty} g(w) e^{-iwt} dw \quad (3)$$

$$g(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau) e^{-i w \tau} d\tau \quad (4)$$

These two equations constitute what is known as
Fourier complex transform pair.

For trigonometric form: Set(frequency); $w_n = \frac{n\pi}{p}$,

$$\Delta w_n = \frac{\pi(n+1)}{p} - \frac{n\pi}{p} = \frac{\pi}{p} \text{ or } \underline{\frac{1}{p} = \frac{\Delta w}{\pi}}$$

$$f_p(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p} \right]$$

$$= \frac{1}{2p} \int_{-p}^p f_p(\tau) d\tau + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\cos(wt) \Delta w \int_{-p}^p f_p(\tau) \right.$$

$$\left. \cos(w\tau) d\tau + \sin(wt) \Delta w \int_{-p}^p f_p(\tau) \sin(w\tau) d\tau \right] (5)$$

$$f_p(t) = \frac{1}{2p} \int_{-p}^p f_p(\tau) d\tau + \frac{1}{\pi} \sum_{n=1}^{\infty} [\cos(wt) \Delta w \int_{-p}^p f_p(\tau) \cos(w\tau) d\tau + \sin(wt) \Delta w \int_{-p}^p f_p(\tau) \sin(w\tau) d\tau] \quad (5)$$

Let $p \rightarrow \infty$ (then $\Delta w \rightarrow 0$) and we can show,

$$f(t) = \frac{1}{\pi} \int_0^{\infty} [\cos wt \int_{-\infty}^{\infty} f(\tau) \cos(w\tau) d\tau + \sin wt \int_{-\infty}^{\infty} f(\tau) \sin(w\tau) d\tau] dw$$

OR:

$$f(t) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(\tau) \cos w\tau \cos wt d\tau dw; \quad (6)$$

f(t) is even [*Fourier cosine integral*]

$$f(t) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(\tau) \sin w\tau \sin wt d\tau dw; \quad (7)$$

f(t) is odd [*Fourier sine integral*]

It is convenient to have the Fourier cosine and sine integral representation as transform pair.

The Fourier cosine transform pair is :

$$\left. \begin{aligned} f(t) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(w) \cos wt \, dw \\ g(w) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos wt \, dt \end{aligned} \right\} f(t) \text{ is even}$$

We have made it instead of τ .

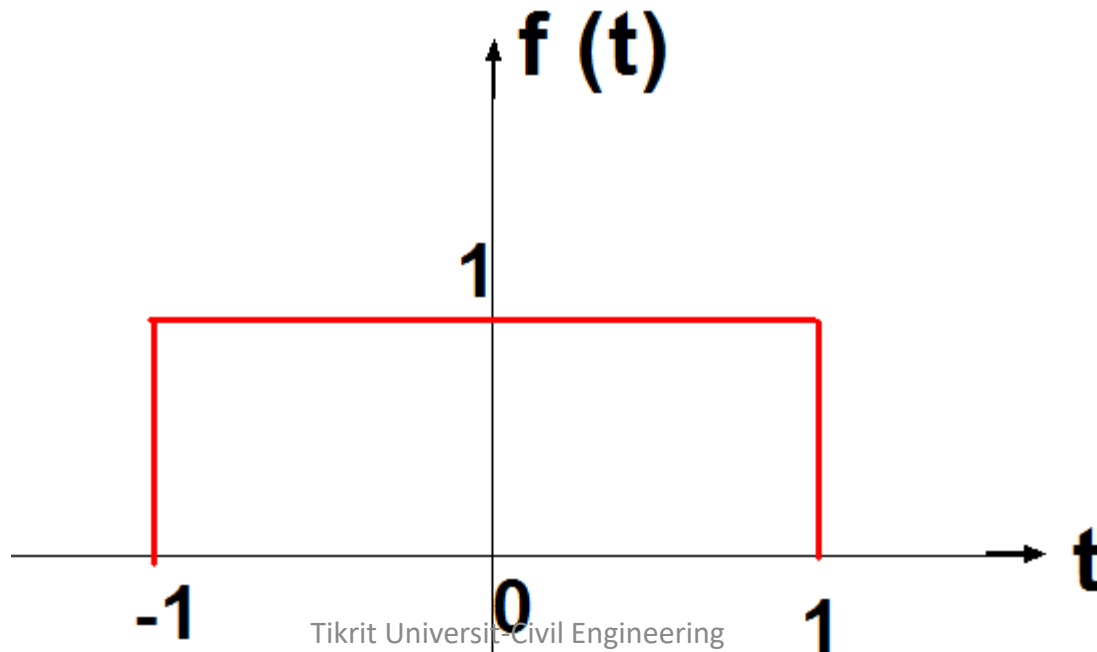
The Fourier sine transform pair is :

$$\left. \begin{aligned} f(t) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(w) \sin wt \, dw \\ g(w) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin wt \, dt \end{aligned} \right\} f(t) \text{ is odd}$$

Example:

Find the Fourier integral of :

$$f(t) = \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$$



Solution:

The function is even, So:

$$g(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos wt \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 1 \cos wt \, dt = \sqrt{\frac{2}{\pi}} \left. \frac{\sin wt}{w} \right|_0^1$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(w) \cos wt \, dw$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\sin w}{w} \cos wt \, dw$$

(a)

The average of the left and right hand limit of :

$f(t)@t=1$ is equal to $\frac{|t_o|}{2} = \frac{1}{2}$

$$\int_0^{\infty} \frac{\cos wt \sin w}{w} dw = \begin{cases} \frac{\pi}{2} & 0 \leq t \leq 1 \\ \frac{\pi}{4} & t = 1 \\ 0 & t > 1 \end{cases}$$

When $t = 0$, then $\int_0^{\infty} \frac{\sin w}{w} dw = Si(\infty) = \frac{\pi}{2}$.

The integral is the limit of the So-called,

sine integral $[Si(t) = \int_0^t \frac{\sin w}{w} dw \text{ as } t \rightarrow \infty.$

In non periodic case, as we did in the Fourier series, it is of interest to investigate how well the Fourier integral represents a function.

Let us return to our example; From Eq.(a) as an approximation to $f(t)$;

we have the finite integral $\frac{2}{\pi} \int_0^{w_o} \frac{\sin w}{w} \cos wt \, dw$ by replacing ∞ by w_o .

And since $\cos a \sin b = \frac{\sin(a + b) - \sin(a - b)}{2}$

$$\therefore \frac{2}{\pi} \int_0^{w_o} \frac{\sin w}{w} \cos wt \, dw$$

$$= \frac{1}{\pi} \int_0^{w_o} \frac{\sin w(t+1)}{w} \, dw - \frac{1}{\pi} \int_0^{w_o} \frac{\sin w(t-1)}{w} \, dw$$

Set $w + wt = u$; then $dw = \frac{du}{t+1}$

$$\frac{dw}{w} = \frac{du}{w(t+1)} = \frac{du}{u} \text{ and } \begin{cases} 0 \leq w \leq w_o \\ 0 \leq u \leq (t+1)w_o \end{cases} \text{ approaches}$$

Also set $w - wt = -u \Rightarrow wt - w = u$

and $0 \leq w \leq w_o$ approaches $0 \leq u \leq (t-1)w_o$

Then we obtain:

$$\begin{aligned} & \frac{2}{\pi} \int_0^{w_o} \frac{\cos wt \sin w}{w} dw \\ &= \frac{1}{\pi} \int_0^{w_o(t+1)} \frac{\sin u}{u} du - \frac{1}{\pi} \int_0^{w_o(t-1)} \frac{\sin u}{u} du \\ &= \frac{1}{\pi} Si[w_o(t+1)] - \frac{1}{\pi} Si[w_o(t-1)] \end{aligned}$$

The figure below shows this approximation for $w_o=4, 8$ and 16 rad/unit time.

Physically speaking, these curves describe the output of an ideal low-pass filter, cutting off all frequencies above w_o , when the input signal is an isolated rectangular pulse.

